

On the correspondence between the solutions of Dirac equation and electromagnetic 4-potentials

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Abstract

In this paper the inverse problem of the correspondence between the solutions of the Dirac equation and electromagnetic 4-potentials, has been completely solved. It is shown that any solution of the Dirac equation with real 4-potential corresponds to one and only one real 4-potential and to infinitely many complex 4-potentials. The expressions of the 4-potentials as functions of the Dirac solution are provided for both cases. It is also proven that every Dirac solution corresponds to one and only one mass of Dirac particles. Some new consistency conditions are also derived.

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1 Introduction

The Dirac equation is one of the greatest achievements of theoretical physics ever made. [1]. It has been the first electron equation in quantum mechanics to satisfy the Lorentz covariance [2], which is an important restriction on physical theories. The introduction of Dirac equation triggered the beginning of one of the most powerful theory ever formulated: the quantum electrodynamics. This equation predicted the spin and the magnetic moment of the electrons, the existence of antiparticles and was able to reproduce accurately the spectrum of the hydrogen atom. Dirac equation has played an important role in various areas of physics such as high energy physics and nuclear physics, while recently it's usefulness has been realized in condensed matter, because the electronic band structure in solids sometimes has features similar to those extracted from Dirac Hamiltonian for massless fermions [3], [4] Despite all the work that has been done

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over the years, Dirac equation is exactly solvable only for very few interactions [5], [6], [7], [8], and the solutions usually come with a strong constraint on the potentials [9], [10], [11].

The majority of the previously reported work focusses in finding the wave function Ψ when the electromagnetic 4- potential is given. An interesting question formulated earlier by Eliezer [12] focused in the inverse problem: “Given the wave function Ψ , what can we say about the electromagnetic potential A_μ , which is connected with Ψ by Dirac’s equation? Is A_μ uniquely determined, and if not, what is the extent to which it is arbitrary? ”. The expressions of A_μ as a function of the components of Ψ allow to be expressed the components of the electromagnetic tensor, $f_{\mu\nu}$, also in the same manner. In his relevant work Eliezer found an expression of the magnetic vector potential components as a function of Ψ and the electric scalar potential ϕ . Eliezer pointed out that the Dirac equation could be written in the matrix form $PX = Q$, where P contains only Dirac spinor components and X contains only electromagnetic potential components. In this case the four equations of the system are not linearly independent, because $\det P = 0$, and so P is not invertible. One important work of Radford in this direction is reported in [13]. In this work the Dirac equation is expressed in 2-spinor form, which allows it to be (covariantly) solved for the electromagnetic 4-potential, in terms of the wave function and its derivatives. This approach subsequently led to some physically interesting results, see also [14], [15], [16] (for a review see [17]). In [18] it was demonstrated that the Dirac equation is indeed algebraically invertible if a real solution for the vector potential is required. Namely, two expressions for the components A_μ of the electromagnetic 4-potential are presented, equivalent to the one given in [13]. These two expressions, by using the standard representation of the Dirac matrices γ_μ , $\mu = 1, 2, 3, 4$, can be equivalently written in the following forms $A_\mu = \frac{1}{2q\Psi^*\gamma_4\Psi}\Phi_\mu$ and $A_\mu = \frac{1}{2q\Psi^*\gamma_1\gamma_2\gamma_3\Psi}\Omega_\mu$, where Φ_μ and Ω_μ are bilinear forms in terms of Ψ and its derivatives. It is obvious that both expressions do not hold in the case $\Psi^*\gamma_4\Psi = \Psi^*\gamma_1\gamma_2\gamma_3\Psi = 0$. Therefore, a question arises here, concerning the existence of nonzero solutions Ψ of the Dirac equation satisfying the conditions $\Psi^*\gamma_4\Psi = \Psi^*\gamma_1\gamma_2\gamma_3\Psi = 0$. It is clear from the above that the inverse problem remains open.

In this work we provide the complete solution to the inverse problem. In section 2 we reproduce the results given in [12], [18], which are needed in the rest of this paper, by conveniently using the standard representation of the Dirac matrices. In section 3 we prove that any non zero solution of Dirac equation connected with a real 4-potential, corresponds to one and only one mass as well as to one and only one real 4-potential. In section 4 we provide a complete solution to the inverse problem, based on the results of the previous two sections. More specifically, we show that any non zero solution Ψ of the Dirac equation with real electromagnetic 4-potential satisfies the condition $\text{supp } \Psi^*(\gamma_4 + \gamma_1\gamma_2\gamma_3)\Psi = \text{supp}(\Psi)$, which is equivalent to either $\Psi^*\gamma_4\Psi \neq 0$ or $\Psi^*\gamma_1\gamma_2\gamma_3\Psi \neq 0$ at any point in $\text{supp}(\Psi)$. Subsequently, we provide the appropriate expressions for the components A_μ of the unique 4- potential in terms

of Ψ and its derivatives, using the results of [18]. Further, we prove that any solution of the Dirac equation with real electromagnetic 4-potential is connected with an infinite number of complex 4-potentials, provided that the condition $\Psi^T \gamma_2 \Psi \neq 0$ holds. Finally, in section 5, some new consistency conditions are presented.

2 Preliminaries

For convenience, we employ the fourth coordinate $x_0 = ct$ and Euclidian metric, which is the simplest physical metric. The Dirac equation which includes an electromagnetic 4- potential [19] can be written in the following form:

$$\left[\sum_{\mu=1}^3 \gamma_{\mu} (\partial_{\mu} - ia_{\mu}) - i\gamma_4 (\partial_0 + ia_0) + \kappa \right] \Psi = 0, \quad (2.1)$$

where $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are the Dirac matrices in the standard representation:

$$\begin{aligned} \gamma_1 &= \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}, & \gamma_2 &= \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \\ \gamma_3 &= \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}, & \gamma_4 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \end{aligned}$$

x_{μ} , $\mu = 0, 1, 2, 3$ are real variables, a_{μ} are real functions of x_{μ} , $\kappa = mc/\hbar$, $\partial_0 = \frac{1}{c}\partial_t$ and Ψ is a 4- component Dirac spinor. More specifically, $a_{\mu} = \frac{e}{\hbar c} A_{\mu}$, $\mu = 1, 2, 3$ with A_{μ} the magnetic vector potential components and $a_0 = e\Phi/\hbar c$ with Φ the electric scalar potential. In the rest of the paper the real constant $\kappa = mc/\hbar$ will be called mass, and this quantity is identical to the inverse of reduced Compton wavelength of Dirac particle with mass m . The Dirac matrices in the standard representation satisfy $\{\gamma_{\mu}, \gamma_{\nu}\} = 2\delta_{\mu\nu}$.

From now on both the 4-potential and the mass will be always real, except where this omission can cause confusion.

For convenience some definitions should be introduced.

Definition 1. Any solution of equation (2.1) for a 4-potential

$$\mathbf{a} := (a_0, a_1, a_2, a_3)$$

and a mass κ will be called Dirac solution.

Definition 2. A Dirac solution Ψ is said to correspond to a mass κ , if there exists a 4-potential \mathbf{a} , such that Ψ is a solution of (2.1) connected with \mathbf{a} and κ .

Definition 3. A Dirac solution Ψ is said to correspond to a 4-potential \mathbf{a} , if there exists a mass κ , such that Ψ is a solution of (2.1) connected with \mathbf{a} and κ .

First, we will show that in addition to the continuity equation

$$\sum_{\mu=1}^3 \partial_{\mu} (\Psi^* \gamma_4 \gamma_{\mu} \Psi) - i \partial_0 (\Psi^* \Psi) = 0 \quad (2.2)$$

there are two additional consistency conditions, presented in [12] and [18]. In the algebraic symbolic of the present work these conditions can be written as follows:

$$\sum_{\mu=1}^3 \partial_{\mu} (\Psi^* \gamma_1 \gamma_2 \gamma_3 \gamma_{\mu} \Psi) - i \partial_0 (\Psi^* \gamma_1 \gamma_2 \gamma_3 \gamma_4 \Psi) + 2\kappa \Psi^* \gamma_1 \gamma_2 \gamma_3 \Psi = 0 \quad (2.3)$$

$$\sum_{\mu=1}^3 \Psi^T \gamma_1 \gamma_3 \gamma_{\mu} \partial_{\mu} \Psi - i \Psi^T \gamma_1 \gamma_3 \gamma_4 \partial_0 \Psi = 0, \quad (2.4)$$

where Ψ^T is the transpose of Ψ . The consistency condition (2.3) is equivalent to (7) in [18], while (2.4) is equivalent to (23) in [12] and (8) in [18].

The first condition (2.3) can be extracted by multiplying (2.1) with $\Psi^* \gamma_1 \gamma_2 \gamma_3$, and subtracting the resulting relation from its Hermitian conjugate. (2.4) can be extracted directly by multiplying (2.1) with $\Psi^T \gamma_1 \gamma_3$ by using that $\gamma_1 \gamma_3$, $\gamma_1 \gamma_3 \gamma_{\mu}$, $\mu = 1, 2, 3, 4$ are antisymmetric.

For convenience we introduce the following notations:

- We denote the following matrices

$$\gamma_5 := \gamma_1 \gamma_2 \gamma_3 \gamma_4, \quad \delta_{\mu} := \gamma_{\mu} + \gamma_5 \gamma_{\mu}, \quad \mu = 1, 2, 3, 4.$$

- Let Ψ be a Dirac solution, and δ is a 4×4 complex or real matrix. If ∂ is a differential operator, then,

$$\overset{\leftrightarrow}{\partial} (\Psi^* \delta \Psi) := \Psi^* \delta \partial \Psi - (\partial \Psi^*) \delta \Psi.$$

In [18] two equivalent algebraic expressions of the 4-potential in terms of the Dirac solution are derived. At this point, we reproduce these expressions, according to the symbols used in this work.

In what follows, the following properties of Dirac matrices will be used: γ_{μ} , $\mu = 1, 2, 3, 4, 5$ are Hermitian, while the matrices $\gamma_{\lambda} \gamma_{\mu}$, $1 \leq \lambda \neq \mu \leq 5$ are anti-Hermitian.

Now, by multiplying (2.1) successively with, Ψ^* and adding the resulting equation with its Hermitian conjugate, $\Psi^* \gamma_1 \gamma_4$ and subtracting the result from

its Hermitian conjugate, $\Psi^* \gamma_2 \gamma_4$ and subtracting the resulting equation from its Hermitian conjugate, $\Psi^* \gamma_3 \gamma_4$ and then subtracting the resulting equation from its Hermitian conjugate, we get respectively,

$$2a_0 (\Psi^* \gamma_4 \Psi) = - \sum_{\mu=1}^3 \partial_\mu (\Psi^* \gamma_\mu \Psi) + i \overleftrightarrow{\partial}_0 (\Psi^* \gamma_4 \Psi) - 2\kappa \Psi^* \Psi. \quad (2.5)$$

$$2ia_1 \Psi^* \gamma_4 \Psi = -2\kappa \Psi^* \gamma_1 \gamma_4 \Psi + \overleftrightarrow{\partial}_1 (\Psi^* \gamma_4 \Psi) - \partial_2 (\Psi^* \gamma_5 \gamma_3 \Psi) + \partial_3 (\Psi^* \gamma_5 \gamma_2 \Psi) + i \partial_0 (\Psi^* \gamma_5 \gamma_1 \Psi), \quad (2.6)$$

$$2ia_2 \Psi^* \gamma_4 \Psi + 2\kappa \Psi^* \gamma_2 \gamma_4 \Psi = \partial_1 (\Psi^* \gamma_5 \gamma_3 \Psi) + \overleftrightarrow{\partial}_2 (\Psi^* \gamma_4 \Psi) - \partial_3 (\Psi^* \gamma_5 \gamma_1 \Psi) + i \partial_0 (\Psi^* \gamma_2 \Psi), \quad (2.7)$$

$$2ia_3 \Psi^* \gamma_4 \Psi + 2\kappa \Psi^* \gamma_3 \gamma_4 \Psi = -\partial_1 (\Psi^* \gamma_5 \gamma_2 \Psi) + \partial_2 (\Psi^* \gamma_5 \gamma_1 \Psi) + \overleftrightarrow{\partial}_3 (\Psi^* \gamma_4 \Psi) + i \partial_0 (\Psi^* \gamma_3 \Psi). \quad (2.8)$$

Further, multiplying (2.1) successively with, $\Psi^* \gamma_5$ and subtracting the result from its Hermitian conjugate, $\Psi^* \gamma_2 \gamma_3$ and adding the result with its Hermitian conjugate, $\Psi^* \gamma_1 \gamma_3$ and adding the resulting equation with its Hermitian conjugate, $\Psi^* \gamma_1 \gamma_2$ and adding the resulting equation with its Hermitian conjugate, respectively we obtain,

$$2a_0 (\Psi^* \gamma_5 \gamma_4 \Psi) = i \overleftrightarrow{\partial}_0 (\Psi^* \gamma_5 \gamma_4 \Psi) - \partial_1 (\Psi^* \gamma_5 \gamma_1 \Psi) - \partial_2 (\Psi^* \gamma_5 \gamma_2 \Psi) - \partial_3 (\Psi^* \gamma_5 \gamma_3 \Psi). \quad (2.9)$$

$$2ia_1 \Psi^* \gamma_5 \gamma_4 \Psi = \overleftrightarrow{\partial}_1 (\Psi^* \gamma_5 \gamma_4 \Psi) - \partial_2 (\Psi^* \gamma_3 \Psi) + \partial_3 (\Psi^* \gamma_2 \Psi) + i \partial_0 (\Psi^* \gamma_5 \gamma_1 \Psi). \quad (2.10)$$

$$2ia_2 \Psi^* \gamma_5 \gamma_4 \Psi = \partial_1 (\Psi^* \gamma_3 \Psi) + \overleftrightarrow{\partial}_2 (\Psi^* \gamma_5 \gamma_4 \Psi) - \partial_3 (\Psi^* \gamma_1 \Psi) + i \partial_0 (\Psi^* \gamma_5 \gamma_2 \Psi), \quad (2.11)$$

$$2ia_3 \Psi^* \gamma_5 \gamma_4 \Psi = -\partial_1 (\Psi^* \gamma_2 \Psi) + \partial_2 (\Psi^* \gamma_1 \Psi) + \overleftrightarrow{\partial}_3 (\Psi^* \gamma_5 \gamma_4 \Psi) + i \partial_0 (\Psi^* \gamma_5 \gamma_3 \Psi), \quad (2.12)$$

The two inversion formulas are given by (2.5), (2.6), (2.7), (2.8) and (2.9), (2.10), (2.11), (2.12) respectively. These inversion formulas stand only in the following case: $\Psi^* \gamma_4 \Psi \neq 0$ or $\Psi^* \gamma_5 \gamma_4 \Psi \neq 0$. Therefore, a question arises here, concerning the existence of nonzero solutions Ψ of the Dirac equation satisfying the conditions $\Psi^* \gamma_4 \Psi = \Psi^* \gamma_1 \gamma_2 \gamma_3 \Psi = 0$. The main aim in the rest of this paper is to answer this question.

3 Uniqueness of mass and 4-potential

In this section we prove that any non zero Dirac solution corresponds to one and only one mass and also to one and only one 4-potential. These results will be used for the certain proofs in the last two sections.

Definition 4. *In the set of all Dirac solutions we define the following relation:*

- $\Psi_1 \approx \Psi_2$ if and only if Ψ_1, Ψ_2 are gauge equivalent, that is there exists a non zero number c and a differentiable function $f : \mathbb{R}^4 \rightarrow \mathbb{R}$ such that $\Psi_1 = ce^{if}\Psi_2$. Clearly " \approx " is an equivalence relation, and by $[\Psi]$ we will denote the equivalence class of Ψ .

For the proofs of the next theorems we need the following lemmas.

Lemma 3.1. *Let Ψ be a non zero Dirac solution. Then any element of $[\Psi]$ is a Dirac solution. If Ψ corresponds to a mass κ , then any element of $[\Psi]$ corresponds to the same mass κ .*

Proof. Since Ψ corresponds to the mass κ there is a 4-potential (a_0, a_1, a_2, a_3) such that Ψ is a solution of (2.1) by these mass and vector field. Let Ψ_1 be any element of $[\Psi]$. Then we have $\Psi = c_1 \exp(if) \Psi_1$, for some non zero $c_1 \in \mathbb{R}$ and some differentiable function $f : \mathbb{R}^4 \rightarrow \mathbb{R}$. So from (2.1) we get

$$\left[\sum_{\mu=1}^3 \gamma_{\mu} (\partial_{\mu} - i(a_{\mu} - \partial_{\mu}f)) - i\gamma_4 (\partial_0 - i(a_0 - \partial_0f)) + \kappa \right] \Psi_1 = 0.$$

Therefore, Ψ_1 corresponds to the mass κ and to the 4-potential

$$(a_0 - \partial_0f, a_1 - \partial_1f, a_2 - \partial_2f, a_3 - \partial_3f).$$

□

Lemma 3.2. *Let Ψ be any Dirac solution. Then there exist at least one $\Psi_0 \in [\Psi]$, such that*

$$-i \overset{\leftrightarrow}{\partial}_0 (\Psi_0^* \gamma_4 \Psi_0) + \sum_{\mu=1}^3 \partial_{\mu} (\Psi_0^* \gamma_{\mu} \Psi_0) + 2\kappa \Psi_0^* \Psi_0 = 0. \quad (3.1)$$

Proof. We define Ψ_0 by

$$\Psi = \exp \left(-i \int_k^{x_0} a_0(s, x_1, x_2, x_3) ds \right) \Psi_0, \quad (3.2)$$

where k is a real constant in the domain of a_0 . Then $\Psi_0 \in [\Psi]$.

Let δ be any matrix with real or complex entries. Then from (3.2) we easily get,

$$i \overset{\leftrightarrow}{\partial}_0 (\Psi^* \delta \Psi) = 2a_0 \Psi_0^* \delta \Psi_0 + i \overset{\leftrightarrow}{\partial}_0 (\Psi_0^* \delta \Psi_0), \quad (3.3)$$

and obviously

$$\Psi^* \delta \Psi = \Psi_0^* \delta \Psi_0, \partial_\mu (\Psi^* \delta \Psi) = \partial_\mu (\Psi_0^* \delta \Psi_0). \quad (3.4)$$

Now, if we substitute (3.2) in (2.5), by using (3.3) and (3.4) we get (3.1). \square

Theorem 3.3. *Any non zero Dirac solution corresponds to one and only one mass.*

Proof. Let Ψ be any Dirac solution. Let $B(\Psi)$ be the set of all masses to which Ψ corresponds. Then, according to Lemma 3.1, we have $B(\Psi_0) = B(\Psi)$ for all $\Psi_0 \in [\Psi]$. Therefore it suffices to show that some element of $[\Psi]$ corresponds to exactly one mass: According to Lemma 3.2 there is one $\Psi_0 \in [\Psi]$, which satisfies (3.1). From $\Psi \neq 0$ it follows that there is one $s \in \mathbb{R}^4$ such that $\Psi^*(s) \Psi(s) \neq 0$ and consequently $\Psi_0^*(s) \Psi_0(s) \neq 0$. Therefore, from (3.1) we get the following unique expression of the mass κ that corresponds to Ψ_0 .

$$\kappa = \frac{i \overset{\leftrightarrow}{\partial}_0 (\Psi_0^*(s) \gamma_4 \Psi_0(s)) - \sum_{\mu=1}^3 \partial_\mu (\Psi_0^*(s) \gamma_\mu \Psi_0(s))}{2 \Psi_0^*(s) \Psi_0(s)}.$$

Therefore Ψ_1 corresponds to a unique mass. \square

Theorem 3.4. *Any non zero Dirac solution corresponds to one and only one 4-potential.*

Proof. Suppose that Ψ corresponds to the following two 4-potentials $\mathbf{a}_i = (a_{0j}, a_{1j}, a_{2j}, a_{3j})$, $j = 1, 2$. Then according to Theorem 3.3 there is unique mass κ such that

$$\left[\sum_{\mu=1}^3 \gamma_\mu (\partial_\mu - i a_{\mu j}) - i \gamma_4 (\partial_0 + i a_{0j}) + \kappa \right] \Psi = 0, \quad j = 1, 2.$$

If we subtract both equations, we get,

$$\left[\gamma_4 (a_{02} - a_{01}) - i \sum_{\mu=1}^3 \gamma_\mu (a_{\mu 2} - a_{\mu 1}) \right] \Psi = 0.$$

Multiplying this last relation by $\gamma_4 (a_{02} - a_{01}) + i \sum_{\mu=1}^3 \gamma_\mu (a_{\mu 2} - a_{\mu 1})$ it follows

$$\left[\sum_{\mu=0}^3 (a_{\mu 2} - a_{\mu 1})^2 \right] \Psi = 0$$

Therefore, the restrictions to $\text{supp}(\Psi)$ of both real 4-potentials $\mathbf{a}_1, \mathbf{a}_2$ are identical. \square

4 The 4-potential in terms of Ψ

In this section the problem of the correspondence between the Dirac solutions and electromagnetic 4-potentials, is completely solved. We will also derive an expression of the 4-potential in terms of the Dirac solution.

Lemma 4.1. *For any $x \in \text{supp}(\Psi)$ we have,*

$$\begin{aligned} (\Psi^T \gamma_2 \Psi) \overline{(\Psi^T \gamma_1 \gamma_2 \gamma_4 \Psi)} + (\Psi^T \gamma_1 \gamma_2 \gamma_4 \Psi) \overline{(\Psi^T \gamma_2 \Psi)} &= 0, \\ (\Psi^T \gamma_2 \Psi) \overline{(\Psi^T \gamma_4 \Psi)} + (\Psi^T \gamma_4 \Psi) \overline{(\Psi^T \gamma_2 \Psi)} &= 0, \\ (\Psi^T \gamma_2 \Psi) \overline{(\Psi^T \gamma_2 \gamma_3 \gamma_4 \Psi)} + (\Psi^T \gamma_2 \gamma_3 \gamma_4 \Psi) \overline{(\Psi^T \gamma_2 \Psi)} &= 0. \end{aligned} \quad (4.1)$$

if and only if

$$(\Psi^T \gamma_2 \Psi) (\Psi^* \delta_4 \Psi) = 0, \quad (4.2)$$

where $\Psi = \Psi(x)$.

Proof. With

$$\Psi = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_1 \\ \zeta_1 \\ \zeta_1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} \zeta_1 \\ \zeta_1 \\ \zeta_1 \\ \zeta_1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \Psi, \quad (4.3)$$

after some algebra, (4.1) can be written as

$$\begin{aligned} (\zeta_1 \zeta_4 - \zeta_2 \zeta_3) \overline{(\zeta_1 \zeta_2 - \zeta_3 \zeta_4)} &= \overline{(\zeta_1 \zeta_4 - \zeta_2 \zeta_3)} (\zeta_1 \zeta_2 - \zeta_3 \zeta_4), \\ (\zeta_1 \zeta_4 - \zeta_2 \zeta_3) \overline{(\zeta_1 \zeta_2 + \zeta_3 \zeta_4)} &= \overline{(\zeta_1 \zeta_4 - \zeta_2 \zeta_3)} (\zeta_1 \zeta_2 + \zeta_3 \zeta_4), \\ \zeta_1 \zeta_4 \zeta_2 \zeta_3 &= \overline{\zeta_1 \zeta_4 \zeta_2 \zeta_3}, \end{aligned}$$

which by

$$\zeta_1 = \omega_1 \zeta_3 \text{ and } \zeta_4 = \omega_2 \zeta_2 \quad (4.4)$$

becomes the following form:

$$\begin{aligned} |\zeta_2 \zeta_3|^2 [(\omega_1 \omega_2 - 1) (\overline{\omega_1} - \overline{\omega_2}) - (\overline{\omega_1 \omega_2} - 1) (\omega_1 - \omega_2)] &= 0, \\ |\zeta_2 \zeta_3|^2 [(\omega_1 \omega_2 - 1) (\overline{\omega_1} + \overline{\omega_2}) + (\overline{\omega_1 \omega_2} - 1) (\omega_1 + \omega_2)] &= 0, \\ |\zeta_2 \zeta_3|^2 (\omega_1 \omega_2 - \overline{\omega_1 \omega_2}) &= 0. \end{aligned}$$

That is

$$\zeta_2 = 0 \text{ or } \zeta_3 = 0 \text{ or } \left\{ \begin{array}{l} (\omega_1 \omega_2 - 1) (\overline{\omega_1} - \overline{\omega_2}) - (\overline{\omega_1 \omega_2} - 1) (\omega_1 - \omega_2) = 0, \\ (\omega_1 \omega_2 - 1) (\overline{\omega_1} + \overline{\omega_2}) + (\overline{\omega_1 \omega_2} - 1) (\omega_1 + \omega_2) = 0 = 0, \\ \omega_1 \omega_2 \in \mathbb{R}, \end{array} \right\}$$

which is equivalent to

$$\zeta_2 = 0 \text{ or } \zeta_3 = 0 \text{ or } \omega_1 \omega_2 - 1 = 0 \text{ or } \left\{ \begin{array}{l} \overline{\omega_1} - \overline{\omega_2} - \omega_1 + \omega_2 = 0, \\ \overline{\omega_1} + \overline{\omega_2} + \omega_1 + \omega_2, \end{array} \right\}$$

or equivalently

$$\zeta_2 = 0 \text{ or } \zeta_3 = 0 \text{ or } \omega_1\omega_2 - 1 = 0 \text{ or } \omega_1 + \overline{\omega_2} = 0,$$

which can be equivalently written as

$$|\zeta_2|^2 \zeta_3^2 (\omega_1\omega_2 - 1) (\omega_1 + \overline{\omega_2}) = 0,$$

or by (4.4)

$$(\zeta_1\zeta_4 - \zeta_2\zeta_3) (\zeta_1\overline{\zeta_2} + \zeta_3\overline{\zeta_4}) = 0,$$

which by using (4.3) can be rewritten as (4.2). \square

Lemma 4.2. *Let U be any non empty open subset of $\text{supp}(\Psi)$. Then for any $x \in \text{supp}(\Psi)$ we have*

$$\Psi^T \gamma_2 \Psi = \Psi^* \delta_4 \Psi = 0, \quad (4.5)$$

if and only if

$$\Psi = [\psi_1 \ \psi_2 \ \psi_1 \ \psi_2]^T \text{ or } [\psi_1 \ \psi_2 \ -\psi_1 \ -\psi_2]^T,$$

where $\Psi = \Psi(x)$ and $\psi_\mu = \psi_\mu(x)$.

Proof. Setting (4.3) in (4.5) we get

$$\zeta_1\zeta_4 = \zeta_2\zeta_3, \ \overline{\zeta_1}\zeta_2 + \overline{\zeta_3}\zeta_4 = 0, \quad (4.6)$$

Suppose that for some $x \in \text{supp}(\Psi)$ holds $\zeta_1\zeta_2\zeta_3\zeta_4 \neq 0$. Then from (4.6) we easily obtain

$$\left| \frac{\zeta_1}{\zeta_3} \right|^2 = -1.$$

Therefore for any $x \in \text{supp}(\Psi)$ holds $\zeta_1\zeta_2\zeta_3\zeta_4 = 0$. Hence, from (4.6) we have that for any $x \in \text{supp}(\Psi)$

$$\zeta_1 = \zeta_3 = 0 \text{ or } \zeta_2 = \zeta_4 = 0$$

which by (4.3) can be rewritten as

$$\psi_4 + \psi_2 = \psi_3 + \psi_1 = 0 \text{ or } \psi_4 - \psi_2 = \psi_3 - \psi_1 = 0.$$

\square

Lemma 4.3. *Let Ψ be a Dirac solution. If $\Psi = [\psi_1 \ \psi_2 \ \psi_1 \ \psi_2]^T$ or $\Psi = [\psi_1 \ \psi_2 \ -\psi_1 \ -\psi_2]^T$ to an open set U , then $\Psi = 0$ to U .*

Proof. If $\Psi = [\psi_1 \ \psi_2 \ -\psi_1 \ -\psi_2]^T$, then from (2.1) we obtain the following system of equations (S),

$$\begin{aligned} i\partial_1\psi_2 + \partial_2\psi_2 + i\partial_3\psi_1 - i\partial_0\psi_1 &= -a_1\psi_2 + ia_2\psi_2 - a_3\psi_1 - a_0\psi_1 - \kappa\psi_1 \\ i\partial_1\psi_1 - \partial_2\psi_1 - i\partial_3\psi_2 - i\partial_0\psi_2 &= -a_1\psi_1 - ia_2\psi_1 + a_3\psi_2 - a_0\psi_2 - \kappa\psi_2 \\ i\partial_1\psi_2 + \partial_2\psi_2 + i\partial_3\psi_1 - i\partial_0\psi_1 &= -a_1\psi_2 + ia_2\psi_2 - a_3\psi_1 - a_0\psi_1 + \kappa\psi_1 \\ i\partial_1\psi_1 - \partial_2\psi_1 - i\partial_3\psi_2 - i\partial_0\psi_2 &= -a_1\psi_1 - ia_2\psi_1 + a_3\psi_2 - a_0\psi_2 + \kappa\psi_2. \end{aligned}$$

We distinguish the following two cases: *Case 1:* $\kappa \neq 0$, then subtracting the first equation from the third, the second from the fourth we obtain respectively $2\kappa\psi_1 = 0$, $2\kappa\psi_2 = 0$. Therefore $\psi_1 = \psi_2 = 0$. *Case 2:* $\kappa = 0$, then (S) obtains the following equivalent form:

$$\begin{aligned} -a_1\psi_2 + ia_2\psi_2 - a_3\psi_1 - a_0\psi_1 &= i\partial_1\psi_2 + \partial_2\psi_2 + i\partial_3\psi_1 - i\partial_0\psi_1 \\ -a_1\psi_1 - ia_2\psi_1 + a_3\psi_2 - a_0\psi_2 &= i\partial_1\psi_1 - \partial_2\psi_1 - i\partial_3\psi_2 - i\partial_0\psi_2, \end{aligned}$$

A trivial solution of (S) is $\Psi = 0$. We suppose, (S) has also a solution $\Psi \neq 0$. Setting $k_i = \text{Re}(\psi_i)$, $l_i = \text{Im}(\psi_i)$, $\text{Re}(c_i) = m_i$, $\text{Im}(c_i) = n_i$, $i = 1, 2$, where $c_1 = i\partial_1\psi_2 + \partial_2\psi_2 + i\partial_3\psi_1 - i\partial_0\psi_1$ and $c_2 = i\partial_1\psi_1 - \partial_2\psi_1 - i\partial_3\psi_2 - i\partial_0\psi_2$, the two equations can be equivalently written as

$$\begin{aligned} -k_1a_0 - k_2a_1 - l_2a_2 - k_1a_3 &= m_1 \\ -l_1a_0 - l_2a_1 + k_2a_2 - l_1a_3 &= n_1 \\ -k_2a_0 - k_1a_1 + l_1a_2 + k_2a_3 &= m_2 \\ -l_2a_0 - l_1a_1 - k_1a_2 + a_3l_2 &= n_2 \end{aligned}$$

At his point we attempt to solve the above real system for the four unknowns a_0, a_1, a_2, a_3 . We may calculate $\det(P) = 0$, where

$$P = \begin{bmatrix} -k_1 & -k_2 & -l_2 & -k_1 \\ -l_1 & -l_2 & k_2 & -l_1 \\ -k_2 & -k_1 & l_1 & k_2 \\ -l_2 & -l_1 & -k_1 & l_2 \end{bmatrix}.$$

Therefore the above system of equations has either no solutions or has a infinite number of solutions, which is impossible according to Theorem 3.4. Hence $\Psi = 0$. \square

If $\Psi = [\psi_1 \ \psi_2 \ \psi_1 \ \psi_2]^t$, in a manner similar as above, we can show that in any case $\Psi = 0$.

Theorem 4.4. *For any Dirac solution Ψ we have*

$$\text{supp}(\Psi^*\delta_4\Psi) = \text{supp}(\Psi),$$

and the restrictions to $\text{supp}(\Psi)$ of the components of the corresponding 4-potential is given by

$$\begin{aligned} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ -ia_0 \end{bmatrix} &= \frac{i}{\Psi^*\delta_4\Psi} \left\{ \kappa \begin{bmatrix} \Psi^*\gamma_1\gamma_4\Psi \\ \Psi^*\gamma_2\gamma_4\Psi \\ \Psi^*\gamma_3\gamma_4\Psi \\ \Psi^*\Psi \end{bmatrix} \right. \\ &\quad \left. - \frac{1}{2} \begin{bmatrix} i\partial_0 & \partial_3 & -\partial_2 & \overset{\leftrightarrow}{\partial_1} \\ -\partial_3 & i\partial_0 & \partial_1 & \overset{\leftrightarrow}{\partial_2} \\ \partial_2 & -\partial_1 & i\partial_0 & \overset{\leftrightarrow}{\partial_3} \\ -\partial_1 & -\partial_2 & -\partial_3 & i\overset{\leftrightarrow}{\partial_0} \end{bmatrix} \begin{bmatrix} \Psi^*\delta_1\Psi \\ \Psi^*\delta_2\Psi \\ \Psi^*\delta_3\Psi \\ \Psi^*\delta_4\Psi \end{bmatrix} \right\}. \quad (4.7) \end{aligned}$$

Proof. If $\Psi = 0$, then $\text{supp}(\Psi^* \delta_4 \Psi) = \text{supp}(\Psi) = \emptyset$. If $\Psi \neq 0$, then $\text{supp}(\Psi^* \delta_4 \Psi) \subseteq \text{supp}(\Psi) \neq \emptyset$. We suppose that the open subsets $\text{supp}(\Psi^* \delta_4 \Psi)$, $\text{supp}(\Psi)$ in \mathbb{R}^4 are not equal. Then there exists a non empty open set $U \subseteq \text{supp}(\Psi)$, such that

$$\Psi^* \delta_4 \Psi|_U = 0. \quad (4.8)$$

Now, if we also suppose that

$$\Psi^T \gamma_2 \Psi|_U = 0, \quad (4.9)$$

then from (4.8), (4.9) and Lemma 4.2 we have that the restriction to U of Ψ has the following form: $\Psi = [\psi_1 \ \psi_2 \ \psi_1 \ \psi_2]^t$ or $[\psi_1 \ \psi_2 \ -\psi_1 \ -\psi_2]^t$. From this and Lemma 4.3 we obtain $\Psi|_U = 0$. This contradicts to the condition $\emptyset \neq U \subseteq \text{supp}(\Psi)$. Therefore there exists a non empty open subset U_1 of U such that $\Psi^T \gamma_2 \Psi \neq 0$ at any point in U_1 . Therefore the functions $\Theta_1, \Theta_2, \Theta_3$ given by

$$\Theta_1 := i \frac{\Psi^T \gamma_1 \gamma_2 \gamma_4 \Psi}{\Psi^T \gamma_2 \Psi}, \quad \Theta_2 := -i \frac{\Psi^T \gamma_4 \Psi}{\Psi^T \gamma_2 \Psi}, \quad \Theta_3 := -i \frac{\Psi^T \gamma_2 \gamma_3 \gamma_4 \Psi}{\Psi^T \gamma_2 \Psi}, \quad (4.10)$$

are well defined on U_1 , and according to Lemma 4.1, are real functions to U_1 . Since Ψ is a Dirac solution, we have that Ψ satisfies the Dirac equation (2.1) for some 4-potential (a_0, a_1, a_2, a_3) and some mass κ . Let f be any real function defined in U_1 . Then (2.1) can be rewritten as

$$\begin{aligned} & \left[\sum_{\mu=1}^3 \gamma_\mu (\partial_\mu - i(a_\mu + f\Theta_\mu)) - i\gamma_4 (\partial_0 + i(a_0 + f)) + \kappa \right] \Psi \\ &= f \left[\gamma_4 - i \sum_{\mu=1}^3 \Theta_\mu \gamma_\mu \right] \Psi \end{aligned}$$

After some algebraic calculations we easily get

$$\left[i \sum_{\mu=1}^3 \Theta_\mu \gamma_\mu - \gamma_4 \right] \Psi = 0.$$

Combining the last two relations we obtain

$$\left[\sum_{\mu=1}^3 \gamma_\mu (\partial_\mu - i(a_\mu + f\Theta_\mu)) - i\gamma_4 (\partial_0 + i(a_0 + f)) + \kappa \right] \Psi = 0. \quad (4.11)$$

Therefore Ψ corresponds to $(a_0 + f, a_1 + f\Theta_1, a_2 + f\Theta_2, a_3 + f\Theta_3)$ for all real functions f defined on U_1 , which contradicts to Theorem 3.4. Hence, the sets $\text{supp}(\Psi^* \delta_4 \Psi)$, $\text{supp}(\Psi)$ are identical.

Now, if we add, (2.5) with (2.9), (2.6) with (2.10), (2.7) with (2.11), and (2.8) with (2.12), we respectively obtain,

$$\begin{aligned} & 2a_0 \Psi^* \delta_4 \Psi + 2\kappa \Psi^* \Psi \\ &= -\partial_1 (\Psi^* \delta_1 \Psi) - \partial_2 (\Psi^* \delta_2 \Psi) - \partial_3 (\Psi^* \delta_3 \Psi) + i \overset{\leftrightarrow}{\partial}_0 (\Psi^* \delta_4 \Psi), \end{aligned} \quad (4.12)$$

$$2ia_1\Psi^*\delta_4\Psi = -2\kappa\Psi^*\gamma_1\gamma_4\Psi + \overset{\leftrightarrow}{\partial}_1(\Psi^*\delta_4\Psi) - \partial_2(\Psi^*\delta_3\Psi) + \partial_3(\Psi^*\delta_2\Psi) + i\partial_0(\Psi^*\delta_1\Psi), \quad (4.13)$$

$$2ia_2\Psi^*\delta_4\Psi = -2\kappa\Psi^*\gamma_2\gamma_4\Psi + \partial_1(\Psi^*\delta_3\Psi) + \overset{\leftrightarrow}{\partial}_2(\Psi^*\delta_4\Psi) - \partial_3(\Psi^*\delta_1\Psi) + i\partial_0(\Psi^*\delta_2\Psi), \quad (4.14)$$

$$2ia_3\Psi^*\delta_4\Psi = -2\kappa\Psi^*\gamma_3\gamma_4\Psi - \partial_1(\Psi^*\delta_2\Psi) + \partial_2(\Psi^*\delta_1\Psi) + \overset{\leftrightarrow}{\partial}_3(\Psi^*\delta_4\Psi) + i\partial_0(\Psi^*\delta_3\Psi). \quad (4.15)$$

Finally, it easy to verify, that the system of equations (4.12), (4.13), (4.14), (4.15) can be rewritten as (4.7). \square

Theorem 4.5. *Let Ψ be a Dirac solution corresponding to a real 4-potential $\mathbf{a} = (a_0, a_1, a_2, a_3)$ and holding the condition $\Psi^T\gamma_2\Psi \neq 0$. Then at least one of the functions $\Theta_1, \Theta_2, \Theta_3$, as given by (4.10), is not real, and Ψ correspond to all complex 4-potentials containing in the set $P(\Psi)$ given by*

$$P(\Psi) = \{(a_0 + f, a_1 + f\Theta_1, a_2 + f\Theta_2, a_3 + f\Theta_3) : f : \text{supp}(\Psi^T\gamma_2\Psi) \rightarrow \mathbb{C}\},$$

and there exists at least a 4-potential in $P(\Psi)$, which is gauge inequivalent to \mathbf{a} .

Proof. If we suppose that the functions $\Theta_1, \Theta_2, \Theta_3$ are all real, then, as we saw in the proof of the Theorem 4.4, see (4.11), Ψ correspond to all real 4-potentials $(a_0 + f, a_1 + f\Theta_1, a_2 + f\Theta_2, a_3 + f\Theta_3)$, where f is any real function defined on $\text{supp}(\Psi^T\gamma_2\Psi)$, which, according to Theorem 3.4, can not be true. Therefore at least one of the functions $\Theta_1, \Theta_2, \Theta_3$ is not real and we can prove that Ψ corresponds to all complex 4-potentials containing in the set $P(\Psi)$, in a manner similar to Theorem 4.4.

We consider the complex 4-potentials $\mathbf{a}_1 = (a_0 + 1, a_1 + \Theta_1, a_2 + \Theta_2, a_3 + \Theta_3)$ and $\mathbf{a}_2 = (a_0 + x_0, a_1 + x_0\Theta_1, a_2 + x_0\Theta_2, a_3 + x_0\Theta_3) \in P(\Psi)$ obtaining by $f = 1$ and $f = x_0$. We suppose that \mathbf{a}_1 and \mathbf{a}_2 are both gauge equivalent to \mathbf{a} . That is the fields $(1, \Theta_1, \Theta_2, \Theta_3), (x_0, x_0\Theta_1, x_0\Theta_2, x_0\Theta_3)$ are conservative. Therefore it holds

$$\partial_0\Theta_\mu = 0, \mu = 1, 2, 3, \quad (4.16)$$

and

$$x_0\partial_0\Theta_\mu + \Theta_\mu = 1, \mu = 1, 2, 3. \quad (4.17)$$

From (4.16) and (4.17) it follows $\Theta_\mu = 1, \mu = 1, 2, 3$, which is equivalent to

$$i\Psi^T\gamma_1\gamma_2\gamma_4\Psi = -i\Psi^T\gamma_4\Psi = -i\Psi^T\gamma_2\gamma_3\gamma_4\Psi = \Psi^T\gamma_2\Psi.$$

Now, after some algebra, we can find the set L , of all solutions Ψ of the above bilinear equations system:

$$L = \left\{ \begin{bmatrix} \psi_1 \\ \psi_2 \\ -\psi_1 \\ -\psi_2 \end{bmatrix}, \begin{bmatrix} \psi_1 \\ 0 \\ \psi_1 \\ 0 \end{bmatrix}, \begin{bmatrix} \psi_1 \\ -\psi_1 \\ -\psi_1 \\ \psi_1 \end{bmatrix}, \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_1 \\ \psi_2 \end{bmatrix}, \begin{bmatrix} 0 \\ \psi_2 \\ 0 \\ -\psi_2 \end{bmatrix}, \begin{bmatrix} \psi_1 \\ \psi_1 \\ \psi_1 \\ \psi_1 \end{bmatrix} \right\}$$

Finally it is easy to verify that all elements of L satisfy identically the equation $\Psi^T \gamma_2 \Psi = 0$, which contradicts to the condition $\Psi^T \gamma_2 \Psi \neq 0$. Therefore at least one of the 4-potentials $\mathbf{a}_1, \mathbf{a}_2$ is gauge inequivalent to \mathbf{a} . \square

5 Consistency Conditions

In this section it is shown that in any class of gauge equivalent Dirac solutions there is one and only one nonempty subclass, such that all elements of this subclass, satisfy some other new bilinear consistency conditions, in addition to (2.2), (2.3), (2.4).

Definition 5. *In the set of all Dirac solutions we define also the following four relations:*

- $\Psi_1 \approx_\mu \Psi_2$, $\mu = 0, 1, 2, 3$ if and only if there exist a non zero number k and a differentiable function $f : \mathbb{R}^4 \rightarrow \mathbb{R}$ with $\partial_\mu f = 0$, such that $\Psi_1 = ke^{if} \Psi_2$. Clearly the four relations " \approx_μ ", $\mu = 0, 1, 2, 3$ are equivalence relations, and by $[\Psi]_\mu$ we will denote respectively the equivalence classes of Ψ .

It is clear that $[\Psi]_\mu \subset [\Psi]$.

Theorem 5.1. *Let Ψ be any Dirac solution. Then there exist $\tilde{\Psi}_0, \tilde{\Psi}_1, \tilde{\Psi}_2, \tilde{\Psi}_3 \in [\Psi]$, such that any $\Psi_0 \in [\tilde{\Psi}_0]_0$, $\Psi_1 \in [\tilde{\Psi}_1]_1$, $\Psi_2 \in [\tilde{\Psi}_2]_2$, $\Psi_3 \in [\tilde{\Psi}_3]_3$ together with the relations (2.2), (2.3), (2.4), also satisfy respectively the following equations,*

$$-i \overset{\leftrightarrow}{\partial}_0 (\Psi_0^* \gamma_4 \Psi_0) + \sum_{\mu=1}^3 \partial_\mu (\Psi_0^* \gamma_\mu \Psi_0) + 2\kappa \Psi_0^* \Psi_0 = 0, \quad (5.1)$$

$$i \overset{\leftrightarrow}{\partial}_0 (\Psi_0^* \gamma_5 \gamma_4 \Psi_0) = \partial_1 (\Psi_0^* \gamma_5 \gamma_1 \Psi_0) + \partial_2 (\Psi_0^* \gamma_5 \gamma_2 \Psi_0) + \partial_3 (\Psi_0^* \gamma_5 \gamma_3 \Psi_0), \quad (5.2)$$

$$\overset{\leftrightarrow}{\partial}_1 (\Psi_1^* \gamma_5 \gamma_4 \Psi_1) - \partial_2 (\Psi_1^* \gamma_3 \Psi_1) + \partial_3 (\Psi_1^* \gamma_2 \Psi_1) + i \partial_0 (\Psi_1^* \gamma_5 \gamma_1 \Psi_1) = 0, \quad (5.3)$$

$$\begin{aligned} & -i \partial_0 (\Psi_1^* \gamma_5 \gamma_1 \Psi_1) + 2\kappa \Psi_1^* \gamma_1 \gamma_4 \Psi_1 \\ & = \overset{\leftrightarrow}{\partial}_1 (\Psi_1^* \gamma_4 \Psi_1) - \partial_2 (\Psi_1^* \gamma_5 \gamma_3 \Psi_1) + \partial_3 (\Psi_1^* \gamma_5 \gamma_2 \Psi_1), \end{aligned} \quad (5.4)$$

$$\partial_1 (\Psi_2^* \gamma_3 \Psi_2) + \overset{\leftrightarrow}{\partial}_2 (\Psi_2^* \gamma_5 \gamma_4 \Psi_2) - \partial_3 (\Psi_2^* \gamma_1 \Psi_2) + i \partial_0 (\Psi_2^* \gamma_5 \gamma_2 \Psi_2) = 0, \quad (5.5)$$

$$\begin{aligned} & 2\kappa \Psi_2^* \gamma_2 \gamma_4 \Psi_2 - i \partial_0 (\Psi_2^* \gamma_2 \Psi_2) \\ & = \partial_1 (\Psi_2^* \gamma_5 \gamma_3 \Psi_2) + \overset{\leftrightarrow}{\partial}_2 (\Psi_2^* \gamma_4 \Psi_2) - \partial_3 (\Psi_2^* \gamma_5 \gamma_1 \Psi_2), \end{aligned} \quad (5.6)$$

$$- \partial_1 (\Psi_3^* \gamma_2 \Psi_3) + \partial_2 (\Psi_3^* \gamma_1 \Psi_3) + \overset{\leftrightarrow}{\partial}_3 (\Psi_3^* \gamma_5 \gamma_4 \Psi_3) + i \partial_0 (\Psi_3^* \gamma_5 \gamma_3 \Psi_3) = 0, \quad (5.7)$$

$$\begin{aligned}
& 2\kappa\Psi_3^*\gamma_3\gamma_4\Psi_3 - i\partial_0(\Psi_3^*\gamma_3\Psi_3) \\
&= -\partial_1(\Psi_3^*\gamma_5\gamma_2\Psi_3) + \partial_2(\Psi_3^*\gamma_5\gamma_1\Psi_3) + \overset{\leftrightarrow}{\partial}_3(\Psi_3^*\gamma_4\Psi_3). \tag{5.8}
\end{aligned}$$

Further, if $\Psi \neq 0$, then the classes $\left[\tilde{\Psi}_\mu\right]_\mu$, $\mu = 0, 1, 2, 3$ are unique.

Proof. We define $\tilde{\Psi}_0$ by

$$\Psi = \exp\left(-i \int_k^{x_0} a_0(s, x_1, x_2, x_3) ds\right) \tilde{\Psi}_0, \tag{5.9}$$

where k is a non zero real constant. Then $\tilde{\Psi}_0 \in [\Psi]$. Therefore according Lemma 3.2 $\tilde{\Psi}_0$ satisfies (5.1). Further if we put (3.3) and (3.4), by $\Psi_0 = \tilde{\Psi}_0$, in (2.9) it follows (5.2).

Let Ψ_0 be any element of $\left[\tilde{\Psi}_0\right]_0$. Then

$$\tilde{\Psi}_0 = c_2 \exp(ig) \Psi_0, \tag{5.10}$$

for some non zero $c_2 \in \mathbb{R}$ and some differentiable function $g : \mathbb{R}^3 \rightarrow \mathbb{R}$, $g = g(x_1, x_2, x_3)$. By substituting (5.10) in (5.1) and (5.2) we conclude immediately that Ψ_0 satisfies the relations (5.1) and (5.2).

Uniqueness of $\left[\tilde{\Psi}_0\right]_0$: If we subtract (5.1) from (5.2) it follows,

$$i\overset{\leftrightarrow}{\partial}_0(\Psi_0^*\delta_4\Psi_0) = 2\kappa\Psi_0^*\Psi_0 + \sum_{\mu=1}^3 \partial_\mu(\Psi_0^*\delta_\mu\Psi_0). \tag{5.11}$$

Let $\hat{\Psi}_0$ be any element of $[\Psi]$ satisfying (5.1), (5.2) and also (5.11). That is

$$i\overset{\leftrightarrow}{\partial}_0(\hat{\Psi}_0^*\delta_4\hat{\Psi}_0) = 2\kappa\hat{\Psi}_0^*\hat{\Psi}_0 + \sum_{\mu=1}^3 \partial_\mu(\hat{\Psi}_0^*\delta_\mu\hat{\Psi}_0). \tag{5.12}$$

Now, from $\Psi_0, \hat{\Psi}_0 \in [\Psi]$ it follows that there exists a non zero $a \in \mathbb{R}$ and a differentiable function $f : \mathbb{R}^4 \rightarrow \mathbb{R}$, such that

$$\hat{\Psi}_0^* = a \exp(if) \Psi_0^*. \tag{5.13}$$

Substituting (5.13) in (5.12), after some calculations, we get,

$$-2\Psi_0^*\delta_4\Psi_0\partial_0f + i\overset{\leftrightarrow}{\partial}_0(\Psi_0^*\delta_4\Psi_0) = 2\kappa\Psi_0^*\Psi_0 + \sum_{\mu=1}^3 \partial_\mu(\Psi_0^*\delta_\mu\Psi_0). \tag{5.14}$$

Subtracting (5.14) from (5.11) we obtain, $(\Psi_0^*\delta_4\Psi_0)\partial_0f = 0$. Therefore, from the condition $\Psi^*\delta_4\Psi \neq 0$ and the fact $\text{supp}(\Psi^*\delta_4\Psi) = \text{supp}(\Psi)$ (see Theorem 5.5) we get $\partial_0f = 0$ to $\text{supp}(\Psi)$, which combined with (5.13) gives $\hat{\Psi}_0^* \in [\Psi_0]_0 = \left[\tilde{\Psi}_0\right]_0$.

Finally, if we define $\tilde{\Psi}_\mu$, $\mu = 1, 2, 3$ by

$$\Psi = \exp \left(i \int_k^{x_\mu} a_\mu(x_0, x_1, x_2, x_3) \Big|_{x_\mu = s} ds \right) \tilde{\Psi}_\mu, \quad (5.15)$$

then, by using

$$\overset{\leftrightarrow}{\partial}_\mu (\Psi^* \delta \Psi) = 2a_\mu \tilde{\Psi}_\mu^* \delta \tilde{\Psi}_0 + i \overset{\leftrightarrow}{\partial}_0 (\tilde{\Psi}_\mu^* \delta \tilde{\Psi}_\mu) \quad (5.16)$$

and

$$\Psi^* \delta \Psi = \tilde{\Psi}_\mu^* \delta \tilde{\Psi}_\mu, \partial_\nu (\Psi^* \delta \Psi) = \partial_\mu (\tilde{\Psi}_\mu^* \delta \tilde{\Psi}_\mu), \quad (5.17)$$

in a manner similar as above, by using (5.15), (5.16), (5.17), we can show that any element $\Psi_1 \in [\tilde{\Psi}_1]_1$, $\Psi_2 \in [\tilde{\Psi}_2]_2$, $\Psi_3 \in [\tilde{\Psi}_3]_3$ satisfies respectively (5.3) and (5.4), (5.5) and (5.6), (5.7) and (5.8) and that the classes $[\tilde{\Psi}_1]_1$, $[\tilde{\Psi}_2]_2$, $[\tilde{\Psi}_3]_3$ are unique. We will omit the details. \square

6 Summary

In the present study it is shown that a solution Ψ of Dirac equation with real 4-potential corresponds to one and only one mass and also to one and only one real electromagnetic 4-potential. Furthermore, any such Dirac solution satisfies the condition $\text{supp } \Psi^* (\gamma_4 + \gamma_1 \gamma_2 \gamma_3) \Psi = \text{supp}(\Psi)$. These results allow us to find an expression of the unique real 4-potential in terms of Ψ . Further, it is shown that any such solution of the Dirac equation corresponds to infinite number of complex 4-potentials and the set of all these 4-potentials is provided.

Additionally, we have derived a number of new consistency conditions, which are satisfied by at least one member in each gauge equivalent class of Dirac solutions. Finally it has been proven that every Dirac solution corresponds to one and only one mass of Dirac particles.

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References

- [1] P.A.M. DIRAC 1928 *The quantum theory of electron* Proc.Roy. Soc. **A117** 610-24.
- [2] DAVID HESTENES 2003 *Spacetime physics with geometric algebra*, American Journal of Physics **71(7)** 691-714.
- [3] HIDETOSHI FUKUYAMA, YUKI FUSEYA, MASAO OGATA, AKITO KOBAYASHI, YOSHIKAZU SUZUMURA 2012 *Dirac electronics in solids*, Physica B: Condensed Matter **407(11)** 1943-47.

- [4] DAVIDM.V. KARASEV 2011 *Graphene as a Quantum Surface with Curvature-Strain Preserving Dynamics* Russian Journal of Mathematical Physics **18(1)** 64-72.
- [5] V.G. BAGROV, D.M. GITMAN 1990 *Exact Solutions of Relativistic Wave Equation* (Kluwer Academic Publishers).
- [6] L. LAM 1970 *Dirac electron in parallel electric and magnetic fields* Physics Letters A **31(7)** 406-06.
- [7] M.M. NIETO AND P.L. TAYLOR 1985 *Solution (Dirac electron in crossed, constant electric and magnetic fields) that has found a problem (relativistic quantized Hall effect)* American Journal of Physics **53(3)** 234-37.
- [8] MD MONIRUZZAMAN AND S.B. FARUQUE 2012 *The exact solution of the Dirac equation with a static magnetic field under the influence of the generalized uncertainty principle* Physics Scripta **85** 035006.
- [9] S. SCHWEBER 1961 *An Introduction to Relativistic Quantum Field Theory* (Row, Peterson and Company, Elmsford, New York).
- [10] BERND THALLER 1992 *The Dirac equation* Texts and Monographs in Physics (Springer-Verlag, Berlin).
- [11] WALTER GREINER 2000 *Relativistic Quantum Mechanics–Wave Equations* (Springer-Verlag, Berlin).
- [12] C.J. ELIEZER 1958 *A Consistency Condition for Electron Wave Functions* Camb. Philos. Soc. Trans. **54 (2)** 247-50.
- [13] C.J. RADFORD 1996 *Localised Solutions of the Dirac-Maxwell Equations* J. Math. Phys. **37(9)** 4418-33.
- [14] H.S. BOOTH, C.J.RADFORD,LIEZER 1997 *The Dirac-Maxwell equations with cylindrical symmetry*, J. Math. Phys. **38 (3)**, 1257-1268.
- [15] C.J. RADFORD, H.S. BOOTH 1999 *Magnetic Monopoles, electric neutrality and the static Maxwell-Dirac equations*, J.Phys.A:Math.Gen. **32**, 5807-5822.
- [16] H.S. BOOTH1998 *The Static Maxwell-Dirac Equations*, PhD Thesis, University of New England.
- [17] H.S. BOOTH *Nonlinear electron solutions and their characteristics at infinity*, The ANZIAM Journal (formerly J.Aust.M.S (B))
- [18] H.S. BOOTH, G. LEGG AND P.D. JARVIS 2001 *Algebraic solution for the vector potential in the Dirac equation* J. Phys. A: Math. Gen. **34** 5667–77.
- [19] SIEGFRIED FLUGGE 1994 *Practical Quantum Mechanics* Vol. II (Springer - Verlag, Berlin).